Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 1

Exercise 1 *(5 scores)*
The Gammafunction Γ is defined by the Lebesgue integral

$$\Gamma(z) = \int_{[0,\infty)} t^{z-1} e^{-t} dt, \quad z > 0.$$

Prove the following assertions:

(a) $\Gamma(z)$ is well-defined for $z > 0$.

(b) Let $z > 0$ and $\varepsilon > 0$. Then the Riemann integral $I_\varepsilon(z) := \int_\varepsilon^{\frac{1}{\varepsilon}} t^{z-1} e^{-t} dt$ and $I(z) = \lim_{\varepsilon \to 0} I_\varepsilon(z)$ exist.

(c) For each $z > 0$ we have $I(z) = \Gamma(z)$.

(d) The Gammafunction satisfies the following formula

$$\Gamma(z+1) = z\Gamma(z), \quad z > 0.$$

(e) The Gammafunction is continuous.

Hint: Use dominated convergence and the relation between Riemann and Lebesgue integrals (see appendix in the script).

Exercise 2 *(4 scores)*
Let $m(dx)$ be the Lebesgue measure on $\mathbb{R}$ and set $\mu(dx) = p(x)m(dx)$ where

$$p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. Show that μ is a probability measure and it holds that

$$\int_{\mathbb{R}} |x|^k \mu(dx) < \infty, \quad \forall k \geq 1.$$

Compute the expressions $\int_{\mathbb{R}} x^k \mu(dx)$ and $\int_{\mathbb{R}} (x-\mu)^k \mu(dx)$ where $k \geq 1$.

Exercise 3 *(4 scores)*
Let X be a random variable such that there exists $A, C > 0$ with

$$\int_\Omega |X(\omega)|^n d\mathbb{P}(\omega) \le AC^n, \quad \forall n \ge 0.$$

Prove that $\mathbb{P}(|X| > C) = 0$.

Exercise 4 *(4 scores)*
Let X be a random variable with $X \ge 0$ a.s. and

$$\mathbb{P}(X \ge n) \ge \frac{1}{n}, \quad \forall n \ge 1.$$

Prove that $\int_\Omega X d\mathbb{P} = \infty$.

Exercise 5 *(4 scores, Talk)*
Prepare a talk on the construction and main properties of the Lebesgue integral.